

On the maximum energy of shock-accelerated cosmic rays at ultra-relativistic shocks

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ABSTRACT

The maximum energy to which cosmic rays can be accelerated at weakly-magnetised ultra-relativistic shocks is investigated. We demonstrate that for such shocks, in which the scattering of energetic particles is mediated exclusively by ion skin-depth scale structures, as might be expected for a Weibel-mediated shock, there is an intrinsic limit on the maximum energy to which particles can be accelerated. This maximum energy is determined from the requirement that particles must be isotropised in the downstream plasma frame before the mean field transports them far downstream, and falls considerably short of what is required to produce ultra-high-energy cosmic rays. To circumvent this limit, a highly disorganised field is required on larger scales. The growth of cosmic-ray induced instabilities on wavelengths much longer than the ion-plasma skin depth, both upstream and downstream of the shock, is considered. While these instabilities may play an important role in magnetic field amplification at relativistic shocks, on scales comparable to the gyroradius of the most energetic particles, the calculated growth-rates have insufficient time to modify the scattering. Since strong modification is a necessary condition for particles in the downstream region to re-cross the shock, in the absence of an alternative scattering mechanism, these results imply that acceleration to higher energies is ruled out. If weakly magnetised ultra-relativistic shocks are disfavoured as high-energy particle accelerators in general, the search for potential sources of ultra-high-energy cosmic rays can be narrowed.

Key words: acceleration of particles – instabilities – plasmas – shockwaves – cosmic rays

1 INTRODUCTION

Ultra-relativistic shocks are known to occur in the outflows of gamma-ray bursts (GRBs), pulsar winds, and active-galactic nuclei (AGN). These non-linear structures are frequently observed to be strong sources of non-thermal radiation, resulting from the inverse-Compton and synchrotron emission of recently accelerated relativistic particles in the local photon and magnetic fields. Gamma-rays and neutrinos produced in hadronic interactions are also expected, although their radiative signatures are more difficult to detect. Ultra-relativistic shocks have also been suggested as potential sources of ultra-high energy cosmic rays (UHECRs, see Bykov et al. 2012, and references therein), where the relativistic Fermi shock-acceleration mechanism is thought to be the primary mechanism for accelerating these particles to energies in excess of 10^{19} eV.

The theory of Fermi acceleration at relativistic shocks

dates back over 30 years (Peacock 1981), and is undoubtedly the most established mechanism for converting the bulk kinetic energy of a relativistic outflow, into non-thermal high-energy particles (see Kirk & Duffy 1999, for a review). The large Lorentz boosts required to transform between the upstream and downstream rest-frames, preclude the usual assumption of near-isotropy, which is frequently used in non-relativistic treatments. In this situation, the details of particle scattering on in situ electromagnetic fluctuations, plays a more important role (see Kirk & Schneider 1987; Lemoine & Revenu 2006; Niemiec et al. 2006, for some examples of studies into the effects). Considerable progress has followed recent particle-in-cell (PIC) simulations, which have succeeded in demonstrating from first principles, that relativistic shocks can accelerate particles through repeated scattering across a shock (e.g. Sironi & Spitkovsky 2011; Stockem et al. 2012). The particle scattering in these simulations is dominated by deflections in short wavelength, plasma skin-depth scale fluctuations ($\lambda \gtrsim c/\omega_{pp}$, where $\omega_{pp} = \sqrt{4\pi ne^2/\bar{\gamma}m_p}$ is the relativistic plasma fre-

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quency, with $\bar{\gamma}$ the mean thermal Lorentz factor), driven by Weibel/two-stream like instabilities (see Sironi et al. 2013, for a thorough numerical investigation). However, if the gyroradius of the particles greatly exceeds the size of these skin-depth scale structures, in the absence of larger scale ($\lambda \gg c/\omega_{pp}$) fluctuations, the particles are tied to the mean field (Casse et al. 2002; Reville et al. 2008), and subsequent crossings from downstream to upstream will be suppressed. Since the mean field, for most realisations, lies in the plane of the shock downstream, as pointed out by Achterberg et al. (2001), unless cross field diffusion is close to the Bohm limit in the downstream region, a particle in the downstream has no chance of overtaking the shock. This requires the field to be dis-organised on scales close to the particles' Larmor radii in the downstream region. Thus, the ability to scatter high energy particles *downstream* of the shock, is the determining factor with regard to maximum energy.

The most obvious recourse for producing fluctuations on the required scales, is to seed them in the upstream via streaming instabilities, initiated by the returning accelerated particles. While several such investigations have been carried out (e.g. Reville et al. 2006; Milosavljević & Nakar 2006; Medvedev & Zakutnyaya 2009), the issue of long-wavelength perturbations relevant to UHECR scattering has not been satisfactorily addressed.

Here we determine the growth of long-wavelength plasma instabilities ($ck \ll \omega_{pp}$) in the limit of ultra-relativistic shock velocities. In particular, we investigate instabilities occurring in shock-precursors, in which the energy density of the returning particle flux, dominates the energy density in the upstream ion rest-frame, which is most likely the case for ultra-relativistic shocks. As we show, the ratio of these energy densities, in the upstream thermal ion frame, is of the order of $e_{cr}/e_{th} \sim \eta \Gamma_{sh}^4$, where η is a measure of the efficiency with which the incoming energy density is reflected as accelerated protons. Current simulations indicate that this fraction is of the order of $\eta \sim 1 - 10\%$ independent of shock Lorentz factor and magnetisation, (below a critical magnetisation $\sigma < \sigma_{crit} \sim 10^{-3}$, Sironi et al. 2013). For shocks with bulk Lorentz factor $\Gamma_{sh} > 10$, the cosmic-ray energy density clearly dominates. This modifies the general plasma conditions in the precursor of a shock propagating into an ambient electron-proton plasma, and must be taken into account when investigating plasma instabilities in the upstream pre-cursors of relativistic shocks.

The outline of the paper is as follows. In the next section, we discuss the scattering and maximum energy expected in the small-angle scattering limit when the mean field component is included. The bulk properties and conditions in the shock-precursor are described in Section 3. In section 4 we use these conditions to determine the dispersion relation for long-wavelength modes $\lambda \gg c/\omega_{pp}$. Section 5 applies these results, in the context of cosmic-ray acceleration. The growth of plasma instabilities in the downstream region is also determined. We conclude with some additional discussion on the implications of these results.

2 PARTICLE SCATTERING AND MAXIMUM ENERGY IN SMALL-SCALE TURBULENCE

We focus on ultra-relativistic ($\Gamma_{sh} \gg 1$) electron-ion shocks, as potential sources of cosmic-ray protons and nuclei. It is convenient when talking about such shocks to introduce the magnetisation parameters in the upstream and downstream regions

$$\sigma_u = \frac{B_u^2}{4\pi n_u m_p c^2}, \quad \sigma_d = \frac{B_d^2}{4\pi \bar{\gamma} n_d m_p c^2}, \quad (1)$$

where all quantities here are measured in the local plasma frame: $B_{u,d}^2/4\pi$ represents the total magnetic pressure in each region, and $\bar{\gamma} \approx \Gamma_{sh}$ is the mean thermal Lorentz factor of the downstream plasma (e.g. Blandford & McKee 1976). We consider the most common ultra-relativistic shock scenario, in which the shock propagates into a plasma with $w_u = e_u + p_u \approx n_u m_p c^2$. Here w_u is the specific enthalpy of the upstream plasma and p_u and e_u are the upstream pressure and energy density respectively. Considering for the moment the mean fields alone, as expected from the ultra-relativistic shock jump conditions, the magnetisation downstream is: a) for parallel shocks, $B_d = B_u$ and $\sigma_d \ll \sigma_u$, or b) for perpendicular shocks, $B_d/B_u = n_d/n_u \approx \bar{\gamma}$ such that $\sigma_d \sim \sigma_u$, with oblique shocks falling in between these two extremes (see Kirk & Duffy 1999, for a review of relativistic MHD jump conditions).

Particle-in-cell simulations of weakly-magnetised ($\sigma_u < 10^{-3}$) relativistic shocks, have demonstrated that non-thermal particle acceleration, is a natural outcome of the shock formation process. For such shocks, the energy dissipation of the upstream plasma is mediated by Weibel or filamentation-type instabilities, resulting in ion skin-depth scale structures that thermalise the incoming flow (Sironi & Spitkovsky 2011; Haugbølle 2011). The self-generated fields, at least on the scale of current simulations, show $\sigma_d \gg \sigma_u$, with typical values of the order $\sigma_d \sim 10^{-2}$, in contrast to the MHD calculations made above.

The scattering of particles on these fluctuations also results in a small fraction, of the order of 1%, escaping back into the upstream, thus providing the injection mechanism in the Fermi acceleration process (Sironi et al. 2013). These particles can scatter further on the Weibel generated structures, and cross the shock multiple times. We consider the relevant energy and length scales involved.

It is assumed that particles have zero probability of escaping infinitely far ahead of the shock, and that transport downstream is the only escape channel (although see below for the implications of this). Hence, in the absence of radiative losses, or finite time limitations, the maximum energy is ultimately determined by the ability to scatter particles *downstream* of the shock. For an ultra-relativistic shock moving along the z -axis with speed β_{sh} , any particles capable of overtaking the shock must have their parallel velocity component $\beta_z > \beta_{sh} \approx 1/3$. Since $\beta_z = \beta \cos \theta \approx \cos \theta$, on transforming the pitch angle into the upstream rest frame, it follows that all particles are confined to a narrow cone of half-opening angle $\theta \sim 1/\Gamma_{sh}$. Once they leave this cone, they are quickly over-taken by the shock. If the pitch angle, $\mu \equiv \cos \theta$, is still closely aligned with the shock normal, ($|\mu| \lesssim \beta_{sh}$), as is typically the case, a Lorentz transformation to the downstream frame results in a reduction of the

cosmic-ray energy $E'_{\text{cr}} = \Gamma_{\text{rel}} E_{\text{cr}} (1 - \beta_{\text{rel}} \beta_{\text{cr}} \mu) \sim E_{\text{cr}} / \Gamma_{\text{sh}}$ where $\Gamma_{\text{rel}} \approx \Gamma_{\text{sh}} / \sqrt{2}$ is the Lorentz factor of the downstream flow measured by an upstream observer. For simplicity, in what follows, we will assume that this is always the case neglecting factors of order unity, i.e. a particle with Lorentz factor γ in the upstream has $\gamma_{\text{d}} = \gamma / \Gamma_{\text{sh}}$ in the downstream, and vice-versa.

For $\sigma_{\text{d}} \gg \sigma_{\text{u}}$, the downstream particle gyro radius in the self-generated field matches the scattering structure scale λ_{d} when

$$\gamma_{\text{d}} = \frac{e B_{\text{d}} \lambda_{\text{d}}}{m_{\text{p}} c^2} = \bar{\gamma} \sigma_{\text{d}}^{1/2} \frac{\lambda_{\text{d}}}{c / \omega_{\text{pp}}} . \quad (2)$$

Particles with Lorentz factor exceeding this value in the downstream frame enter the small-angle scattering regime, which allows them to diffuse back across the shock. Recalling that simulations suggest $\sigma_{\text{d}} \sim 10^{-2}$ and $\lambda \sim 10 - 20 c / \omega_{\text{pp}}$, it is clear that even mildly supra-thermal particles have sufficient energy to enter this scattering regime, as is indeed observed in PIC simulations. However, as the particles are accelerated to much larger energies, these small-angle deflections ultimately become insignificant, and the mean field must again be considered. Unless some additional scattering field exists at longer wavelengths, particles will simply gyrate in the large-scale mean fields, which recedes at a velocity $\approx c/3$ in the downstream. Cross field diffusion close to the Bohm limit in the downstream region is required for a particle to have any chance of returning to the shock (Achterberg et al. 2001). This requires fluctuations to be generated at, or close to, the scale of particles' Larmor radii in the downstream region.

Following Kirk & Reville (2010), we quantify the small angle scattering behaviour in terms of an angular diffusion coefficient, $D_{\theta} = \langle \Delta \theta^2 \rangle / 2\tau$ where τ is the mean time between scatterings, and $\Delta \theta$ the average deflection angle. For ultra-relativistic particles, the scattering time is approximately the light-crossing time of the Weibel-generated structures. The average deflection angle at each scattering, for a particle with Lorentz factor γ_{d} , in the downstream region is

$$\Delta \theta_{\text{d}}(\gamma_{\text{d}}) \approx \frac{e \sqrt{\delta B_{\text{d}}^2} \lambda_{\text{d}}}{\gamma_{\text{d}} m_{\text{p}} c^2} = \frac{\bar{\gamma}}{\gamma_{\text{d}}} \frac{\lambda_{\text{d}}}{c / \omega_{\text{pp}}} \sigma_{\text{d}}^{1/2} , \quad (3)$$

from which we can evaluate

$$D_{\theta} \approx \left(\frac{\bar{\gamma}}{\gamma_{\text{d}}} \right)^2 \frac{\lambda_{\text{d}}}{c / \omega_{\text{pp}}} \sigma_{\text{d}} \omega_{\text{pp}} . \quad (4)$$

So far, we have neglected the effect of the mean field in the downstream. The mean isotropisation time for a distribution of particles is $\sim D_{\theta}^{-1}$, which becomes increasingly large for $\gamma_{\text{d}} \gg \bar{\gamma}$. Above a critical energy the particles will again return to quasi-helical orbits. This can be expected to occur when the isotropisation time exceeds the Larmor period in the shock compressed mean field, e.g. Lemoine & Pelletier (2010)

$$D_{\theta} \Omega_{\text{g}}^{-1} = D_{\theta} \frac{\gamma_{\text{d}} m_{\text{p}} c}{e \langle B_{\text{d}} \rangle} < 1 . \quad (5)$$

Hence, in the absence of larger scale fluctuations close to spatial resonance with cosmic-ray Larmor radius in the downstream region, the maximum Lorentz factor is limited to

$$\gamma_{\text{d,max}} \sim \bar{\gamma} \frac{\lambda_{\text{d}}}{c / \omega_{\text{pp}}} \sigma_{\text{d}} \sigma_{\text{u}}^{-1/2} \quad (6)$$

as measured in the downstream frame, with σ_{u} the magnetisation far upstream. Lemoine & Pelletier (2010) arrive at the same limiting energy. If particles at this energy can somehow escape upstream, they will gain another factor $\bar{\gamma}$, suggesting a maximum energy

$$E_{\text{max}} \approx \left(\frac{\Gamma_{\text{sh}}}{100} \right)^2 \left(\frac{\lambda_{\text{d}}}{10 c / \omega_{\text{pp}}} \right) \left(\frac{\sigma_{\text{d}}}{10^{-2}} \right) \left(\frac{\sigma_{\text{u}}}{10^{-8}} \right)^{-1/2} \text{ PeV} , \quad (7)$$

where we have chosen parameters that might be relevant for an external GRB shock. In what follows, we investigate the possibility for generating magnetic fluctuations on sufficient scale and amplitude to facilitate acceleration to higher energies than the above limit. Finally, we note that a distant observer will measure the downstream energy, boosted by the relative Lorentz factor $\gamma_{\text{obs}} = \Gamma_{\text{rel}} \gamma_{\text{d}}$. However, for the sources of interest, by the time the particles are released into the interstellar or intergalactic medium (ISM/IGM), the flow will have decelerated appreciably, in which case the maximum energy will be reduced via adiabatic losses by a factor $\sim \bar{\gamma}$. Hence particle escape is an important factor, for accelerating to high energies.

3 REFLECTED PARTICLES AND ELECTRON DRIFT

As mentioned in the previous section, using the MHD shock jump conditions, the mean thermal energy per particle, resulting from the thermalisation of a cold incoming fluid, as measured in the downstream plasma frame is $\bar{\gamma} \approx \Gamma_{\text{sh}}$. It is assumed given that a small fraction of these shocked particles can escape back into the upstream. The mean energy for such particles, now measured in the upstream rest frame, is $\gamma_{\text{u}} \approx \Gamma_{\text{sh}}^2$. For simplicity, we assume that the particle mean-free path is longer than the shock transition layer, which need not, and for low-energy cosmic rays in a Weibel-mediated shock, most likely is not, the case. Any modification however, is expected to be small, and decreases with increasing particle energy. As mentioned in the previous section, a Lorentz transformation from upstream to downstream results, on average, in a reduction of a cosmic-ray's energy, although the over-all energy gain per cycle, is still a net gain. The particles energy is of course approximately constant as measured by a distant observer, however, if the particle is adiabatically coupled to the downstream flow, it will lose energy as the flow decelerates before finally releasing particles. With regards UHECR acceleration, this can have a significant effect on the maximum energy that can be achieved at a shock, since the initial Γ^2 increase is effectively wasted unless the particle can ultimately escape to infinity, upstream of the shock, before it has decelerated appreciably.

Here, the possible contribution of relativistic shocks to the cosmic-ray production is addressed. We focus, initially, on the first generation of shock reflected particles, i.e. those that, as measured in the upstream rest frame, have Lorentz factor $\Gamma_{\text{cr}} \approx \Gamma_{\text{sh}}^2$, and for simplicity consider a pure electron-proton shock. In the shock rest frame, the far upstream plasma can be treated as cold beam, with energy density $\approx \Gamma_{\text{sh}} \bar{n}_{\text{p}} m_{\text{p}} c^2$, where \bar{n}_{p} is the ambient proton density measured in the shock frame. If a fraction η of this energy

leaks back into the upstream¹ with average velocity $-v_{\text{sh}}$ in the shock frame, on transforming back to the upstream proton frame, this reflected component has energy density $\approx 4\eta\Gamma_{\text{sh}}^4 n_{\text{p}} m_{\text{p}} c^2$, i.e. if $\Gamma_{\text{cr}} \approx \Gamma_{\text{sh}}^2$, the cosmic-ray density, as measured in the proton frame is $n_{\text{cr}} \approx 4\eta\Gamma_{\text{sh}}^2 n_{\text{p}}$. If, as simulations suggest, $\eta \sim 1\%$, for $\Gamma_{\text{sh}} \gg 1$, the number density of the cosmic rays can easily exceed that of the background in this frame. As we demonstrate, this can considerably alter the equilibrium plasma conditions in the upstream. This effect, is not confined to the lowest energy particles. The efficiency factor for returning cosmic rays is the sum over all energetic particles $\eta \propto \int_{\gamma_{\text{min}}}^{\infty} n(\gamma) d\gamma$. Far upstream, where only higher energy particles can reach, γ_{min} will increase, and unless $n(\gamma)$ is very flat, $\eta\Gamma_{\text{sh}}^2$ will decrease upstream. Theory and simulations suggest $n(\gamma) \propto \gamma^{-2.2}$ (e.g. Kirk et al. 2000; Achterberg et al. 2001; Sironi et al. 2013), such that η will decrease slightly faster than γ_{min}^{-1} . Hence, beyond a critical distance upstream $\eta\Gamma_{\text{sh}}^2 < 1$, and the current density is too low to have a significant effect. In what follows, unless otherwise stated, we will take $\eta\Gamma_{\text{sh}}^2 \gg 1$ to hold at all times.

We recall that the conditions of charge and current neutrality are,

$$n_{\text{e}} = n_{\text{p}} + n_{\text{cr}} \quad \text{and} \quad n_{\text{e}}\beta_{\text{e}} = n_{\text{p}}\beta_{\text{p}} + n_{\text{cr}}\beta_{\text{cr}} \quad (8)$$

which, provided all terms are appropriately defined, is frame independent. Considering initially, the ambient protons' rest frame, the electrons provide a return current to satisfy the above conditions. Using the above value for n_{cr} , the electrons will drift with velocity

$$\beta_{\text{e}} = \frac{4\eta\Gamma_{\text{sh}}^2}{1 + 4\eta\Gamma_{\text{sh}}^2} \beta_{\text{cr}} \quad (9)$$

with respect to the protons, to cancel the cosmic-ray current.

Since the electric field approximately vanishes in the electrons' rest-frame, it is advantageous to work in this frame. To prevent confusion, we use upper case Γ for quantities measured in the upstream proton frame, and lower case γ for quantities measured in electron frame. From (9) it follows that the protons, by symmetry, now drift with respect to the electrons with Lorentz factor

$$\gamma_{\text{p}} \approx \sqrt{2\eta}\Gamma_{\text{sh}} \quad , \quad (10)$$

while the Lorentz factor of the shock relative to the electrons is significantly reduced:

$$\gamma_{\text{sh}} \approx \frac{1}{\sqrt{8\eta}} \quad . \quad (11)$$

The cosmic rays, which had Lorentz factor Γ_{cr} in the proton frame, now have

$$\gamma_{\text{cr}} \approx \frac{\Gamma_{\text{cr}}}{\sqrt{8\eta}\Gamma_{\text{sh}}} \quad . \quad (12)$$

Since both $\gamma_{\text{p}}, \gamma_{\text{cr}} \gg 1$ under our assumption, it follows from the charge and current conditions, that $n_{\text{e}} \approx 2n_{\text{p}} \approx 2n_{\text{cr}}$ in the electron rest frame.

It is also necessary to consider the equilibrium configuration of the magnetic field in this limit. Since, to lowest

order, the magnetic field is expected to be frozen into the electrons, the net drift of electrons with respect to the protons can have a significant impact, even without resorting to plasma instabilities. Sufficiently far upstream, beyond the range of influence of the cosmic rays, there is no drift between the protons and electrons, and the electric field is approximately zero in the plasma rest-frame. As the shock approaches, the cosmic-rays cause the electrons to drift with respect to protons. From charge conservation

$$\frac{\partial n_{\text{e}}}{\partial t} + \nabla \cdot (n_{\text{e}} \mathbf{v}_{\text{e}}) = 0 \quad , \quad (13)$$

and the magnetic induction equation,

$$\frac{\partial \mathbf{B}}{\partial t} = -c \nabla \times \mathbf{E} \approx \nabla \times (\mathbf{v}_{\text{e}} \times \mathbf{B}) \quad , \quad (14)$$

for flow in one dimension, it follows that $B_{\perp}/n_{\text{e}} = \text{const}$. Inside the shock precursor, in the rest frame of the electrons, the electric field remains zero, and the density is $n_{\text{e}} = 2n_{\text{p}} = 2\sqrt{2\eta}\Gamma_{\text{sh}} n_{\text{p},0}$, where $n_{\text{p},0}$ is the proton proper density, which is also the electron density far upstream. Hence, the perpendicular component of the magnetic field in the rest frame of the compressed electrons, is $B_{\perp} = 2\sqrt{2\eta}\Gamma_{\text{sh}} B_{\perp}^0$. Note that in the shock frame, the fields are still $\hat{B}_{\perp} = \Gamma_{\text{sh}} B_{\perp}^0$, and $\hat{E}_{\perp} = -\beta_{\text{sh}} \times \hat{B}_{\perp}$, such that the downstream fields still satisfy the Rankine-Hugoniot relations for a single fluid weakly magnetised MHD shock (e.g. Kennel & Coroniti 1984).

Finally we note that, as mentioned above, higher energy cosmic-rays extend further upstream from the shock than lower energy ones. Since the efficiency factor depends on all the cosmic-rays that contribute to the returning flux, i.e. η is a function of γ_{min} , it will also have a spatial dependence. The cosmic-ray number density gradually decreases upstream, eventually to the level where the field is unmodified. The above effect will produce a large-scale inhomogeneous magnetic field, with associated current $\mathbf{j}_{\perp} = \nabla \times \mathbf{B}_{\perp}$. The precise details of this current will depend on the shape of the spectrum, as well as the scattering upstream. However, since $j_{\perp} \ll j_{\text{cr}}$ we ignore it in the following.

4 LINEAR STABILITY ANALYSIS OF FORESHOCK REGION

We continue to work in the electron drift frame. Using the conditions described in the previous section, i.e. equations (10) – (12), we investigate the evolution of linear perturbations to the background plasma $\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1$ etc. This is not a straightforward task for relativistic shocks, since it is not immediately clear how to construct an equilibrium solution. As demonstrated in the previous section, unless the ambient field and shock normal are aligned to within $1/\sqrt{8\eta}\Gamma_{\text{sh}}$, the field will be highly oblique, and a zeroth order $\mathbf{v}_0 \times \mathbf{B}_0$ force will act on each species that drifts with respect to the electrons.

To make progress, we approximate the cosmic rays as infinitely rigid ($n_{\text{cr},1} = v_{\text{cr},1} = 0$), and the electrons as a massless fluid. The protons will still accelerate on account of the $\mathbf{v}_0 \times \mathbf{B}_0$ force, and generate an oblique current, which will be neutralised by the massless electrons. Hence, the electron frame must itself accelerate. This effect can be accounted for with the inclusion of a gravity term in the proton equation of motion.

¹ The incident energy flux of the upstream thermal plasma, measured in the shock frame is $T_{\text{in}}^0 = \Gamma_{\text{sh}}^2 \beta_{\text{sh}} n_0 m_{\text{p}} c^2$, where n_0 is the upstream proper density. We define η as the fraction of this energy flux that is reflected $T_{\text{ref}}^0 = -\eta\Gamma_{\text{sh}}^2 \beta_{\text{sh}} n_0 m_{\text{p}} c^2$.

Both the electrons and protons satisfy charge conservation

$$\frac{\partial n_\alpha}{\partial t} + \nabla \cdot (n_\alpha \beta_\alpha c) = 0 \quad (15)$$

where β_α is the fluid velocity of each species in units of c , and both species are treated as cold fluids

$$\frac{d\mathbf{p}_\alpha}{dt} = q_\alpha (\mathbf{E} + \beta_\alpha \times \mathbf{B}) + \gamma_\alpha m_\alpha \mathbf{g} \quad (16)$$

The gravity term \mathbf{g} , representing the acceleration of the non-inertial electron frame, by definition acts to cancel the zeroth-order $\beta_{p,0} \times \mathbf{B}_0$ force, should it exist. Since the electrons are massless, $d\mathbf{p}_e/dt = m_e \mathbf{g} = 0$ and the electric field is $\mathbf{E} = -\beta_e \times \mathbf{B}_0$, where β_e is a first order quantity.

Since $\beta_e \ll 1$, we can safely neglect the displacement current in Ampere's law,

$$\nabla \times \mathbf{B} = \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} + 4\pi \sum_{\alpha=p,e} n_\alpha q_\alpha \beta_\alpha, \quad (17)$$

such that the electron velocity can be expressed in terms of the other perturbed quantities:

$$\beta_e = \frac{n_{p0}}{n_{e0}} \left[\mathbf{v}_1 + \frac{n_{p1}}{n_{p0}} \mathbf{v}_0 - \frac{c\Omega}{\omega_{pp}^2} \nabla \times \mathbf{b}_1 \right], \quad (18)$$

where $\beta_p = \mathbf{v}_0 + \mathbf{v}_1$ is the proton velocity, $\Omega = eB_0/\gamma_0 m_p c$ the relativistic proton gyro frequency, $\omega_{pp} = (4\pi n_p e^2/\gamma_0 m_p)^{1/2}$ the proton plasma frequency, and $\mathbf{b}_1 = \mathbf{B}_1/B_0$ the normalised magnetic field fluctuations. As shown in the previous section, in the ultra-relativistic limit, we have $n_{e0} = 2n_{p0} = 2n_{cr}$, and the above becomes

$$\beta_e = \frac{1}{2} \mathbf{v}_1 + \frac{1}{2} \frac{n_{p1}}{n_{p0}} \mathbf{v}_0 - \frac{1}{2} \frac{c\Omega}{\omega_{pp}^2} \nabla \times \mathbf{b}_1. \quad (19)$$

Together with equations (14) and (15), it can be shown that

$$\begin{aligned} \frac{d}{dt} \left[\frac{\partial \mathbf{b}_1}{\partial t} + \frac{c^2}{2} \frac{\Omega}{\omega_{pp}^2} (\mathbf{b}_0 \cdot \nabla) (\nabla \times \mathbf{b}_1) \right] = \\ \frac{c}{2} \left[(\mathbf{b}_0 \cdot \nabla) \left(\frac{d\mathbf{v}_1}{dt} - c\mathbf{v}_0 (\nabla \cdot \mathbf{v}_1) \right) - \mathbf{b}_0 \frac{\partial}{\partial t} (\nabla \cdot \mathbf{v}_1) \right], \end{aligned} \quad (20)$$

where the convective derivative is with respect to the zeroth order proton velocity $d/dt = \partial/\partial t + c\mathbf{v}_0 \cdot \nabla$. Similarly, linearising the momentum equation produces

$$\begin{aligned} \frac{\partial}{\partial t} \frac{d}{dt} [\mathbf{v}_1 + \gamma_0^2 (\mathbf{v}_1 \cdot \mathbf{v}_0) \mathbf{v}_0] = \\ \frac{\Omega}{2} \left[\frac{\partial \mathbf{v}_1}{\partial t} \times \mathbf{b}_0 + c(\mathbf{b}_0 \cdot \nabla) \mathbf{v}_0 \times \mathbf{v}_1 \right] - \\ \frac{c}{2} \frac{\Omega^2}{\omega_{pp}^2} \left[\mathbf{b}_0 \times \left(\nabla \times \frac{\partial \mathbf{b}_1}{\partial t} \right) + \right. \\ \left. c(\mathbf{b}_0 \cdot \nabla) \mathbf{v}_0 \times (\nabla \times \mathbf{b}_1) \right]. \end{aligned} \quad (21)$$

While these equations are closed, and can be used to derive a complete dispersion tensor, the result is cumbersome and is too complicated to provide the desired insight. However, since we are only concerned with long-wavelength fluctuations, we can simplify the problem considerably. For characteristic time scales ω^{-1} and length scales k^{-1} , the ordering of the terms on the right hand side of equation (21) is

$$\frac{\omega}{\Omega} : \frac{kc}{\Omega} : \frac{kc}{\omega_{pp}} \frac{\omega}{\omega_{pp}} : \frac{c^2 k^2}{\omega_{pp}^2}$$

such that for long-wavelength fluctuations ($\gg c/\omega_{pp}$) the final two terms on the right involving the fluctuating magnetic field can be neglected, and the remaining terms

$$\begin{aligned} \frac{\partial}{\partial t} \frac{d}{dt} [\mathbf{v}_1 + \gamma_0^2 (\mathbf{v}_1 \cdot \mathbf{v}_0) \mathbf{v}_0] = \\ \frac{\Omega}{2} \left[\frac{\partial \mathbf{v}_1}{\partial t} \times \mathbf{b}_0 + c(\mathbf{b}_0 \cdot \nabla) \mathbf{v}_0 \times \mathbf{v}_1 \right] \end{aligned} \quad (22)$$

contain only the different components of \mathbf{v}_1 , which can be solved. We consider the dispersion of plane-waves satisfying equation (22) for the two limiting cases of exactly parallel and perpendicular shocks.

4.1 Parallel shock

Although parallel ultra-relativistic shocks, i.e. those for which the magnetic field is aligned with the shock normal to within $1/\Gamma_{sh}$, are likely to be very rare in Nature, (assuming a random distribution of shock propagation directions and ambient field orientations), they have nevertheless been, by far, the most commonly studied case (e.g. Milosavljević & Nakar 2006; Reville et al. 2006). We use our formalism to investigate the behaviour of long-wavelength linear fluctuations at such shocks. Parallel shocks represent a singular case for relativistic shocks, since reflected particles can in principle propagate upstream for large distances, provided the upstream medium is sufficiently uniform. Similar to supernova remnant shocks, the particles are expected to be self confined due to self-generated turbulence (e.g. Bell et al. 2013).

We look for plane wave solutions to equation (22), $\mathbf{v}_1 = \tilde{\mathbf{v}} \exp[i(\mathbf{k} \cdot \mathbf{x} - \omega t)]$. Taking the shock normal in the negative z direction, i.e. $v_0 > 0$, the dispersion relation reads

$$\omega^2 (\omega - ck_z v_0)^2 - \frac{\Omega^2}{4} (\omega + ck_z v_0)^2 = 0 \quad (23)$$

which, provided $\mathbf{k} \cdot \mathbf{B}_0 \neq 0$, has always one unstable mode. The instability results from the uncompensated currents associated with the thermal background, which is maximised for perturbations along the mean field, i.e. $k = k_z$. The frequency for the growing mode is

$$\omega = \frac{1}{2} \left(ck_z v_0 - \frac{\Omega}{2} \right) + \sqrt{\left(\frac{ck_z v_0}{2} - \frac{\Omega}{4} \right)^2 - \frac{\Omega ck_z v_0}{2}} \quad (24)$$

which has maximum growth rate and corresponding wavenumber

$$\omega_{\max} = \left(\frac{1}{2} + \frac{1}{\sqrt{2}} i \right) \Omega \text{ and } k_{\max} = \frac{3}{2} \frac{\Omega}{cv_0} \quad (25)$$

Thus the maximum growth occurs on scales close to the gyro-radius of the background drifting protons, which is smaller than that of the cosmic-rays by a factor $\sim \Gamma_{cr}/\eta \Gamma_{sh}^2$.

Finally we note that this result, for exactly parallel shocks, can be reproduced using a more detailed kinetic analysis, which includes electron inertia, displacement current, finite temperature effects, as well as feedback on the cosmic rays. This result is given in the appendix, following the approach of Reville et al. (2006). Figures 1 and 2 show a comparison of the dispersion relation, as given in equation (24), with the numerical solutions of equation (A7). Both the real and imaginary parts of the dispersion relation are well approximated in the regime of interest, demonstrating

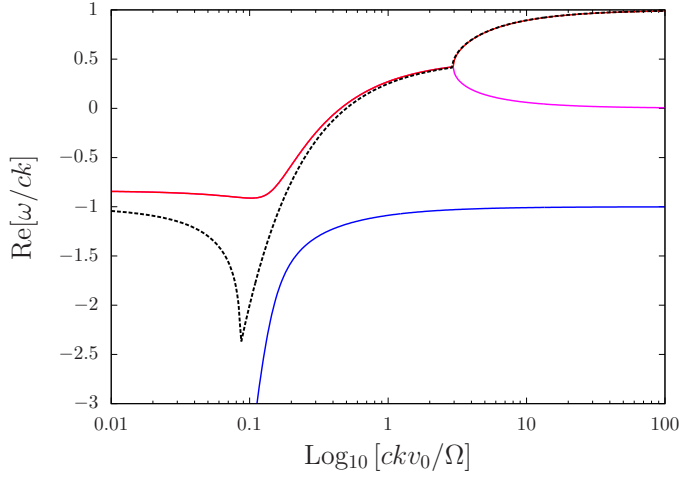


Figure 1. Real part of the dispersion relation (dashed line) from equation (24), for a parallel shock. For comparison the solid lines are the solutions to the dispersion relation for the more detailed model given in the appendix. The lines overlap in the range of interest. The shock Lorentz factor (measured in the proton frame) is $\Gamma_{\text{sh}} = 105$, $\eta = 0.022$, and $\Omega^2/\omega_{\text{pp}}^2 = 2 \times 10^{-5}$.

that the assumptions made in the derivation of the dispersion relation are acceptable.

4.2 Perpendicular shock

The more pertinent case is that of a purely perpendicular shock. Certainly in the frame of the shock, the field is very close to perpendicular, due to the Lorentz transform of the perpendicular magnetic field component, unless it is closely aligned with the shock normal. As we have shown in the previous section, the perpendicular component of the field is also compressed due to the drifting of the electrons to compensate the cosmic-ray current. Previous investigations of plasma instabilities operating at perpendicular shocks have worked in the limit of unmagnetised particles (e.g. Lemoine & Pelletier 2010; Shaisultanov et al. 2012) which can not access the scales of interest here. Pelletier et al. (2009) carried out a single-fluid MHD plus cosmic-ray analysis for magnetised shocks, however, as shown in section 3, a single fluid MHD treatment is no longer viable in the ultra-relativistic limit.

We consider the shock normal again in the negative z -directions, with magnetic field in the y -direction. The acceleration of the electron frame in this case is $\mathbf{g} = cv_0\Omega\hat{\mathbf{x}}$, and all modes are measured in this non-inertial frame. Looking for plane wave solutions to equation (22), we find

$$\omega^2(\omega - ck_z v_0)^2 - \omega^2 \left(\frac{\Omega}{2\gamma_0} \right)^2 - \left(\frac{\Omega}{2} ck_y v_0 \right)^2 = 0. \quad (26)$$

Clearly in the limit of $k_y = 0$ all modes are stable. Unstable solutions can be admitted however if $k_y \neq 0$. We focus on $k = k_y$ modes, in which case calculating the dispersion relation is straightforward. Unstable modes are found for all k in our long-wavelength approximation, although the instability will be suppressed at large k when magnetic tension becomes important. At these scales, however, the particles are all unmagnetised, and the ion-Weibel instability

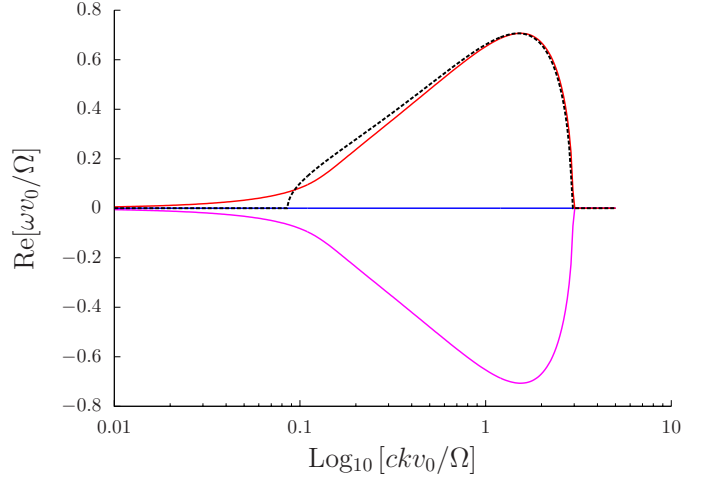


Figure 2. Comparison of the growth rates for a parallel shock, given by Equation (24) (dashed line), with those found using a fully kinetic treatment (solid lines), for the same conditions as in Figure 1

will dominate (e.g. Shaisultanov et al. 2012). The instability is, in our approximation, almost purely growing at all wavelengths, and can be separated into two distinct regimes

$$\text{Im}(\omega) = \begin{cases} \frac{\gamma_0 ck_y v_0}{\sqrt{\Omega ck_y v_0/2}} & ckv_0 \ll \Omega/4\gamma_0^2 \\ \sqrt{\Omega ck_y v_0/2} & ckv_0 \gg \Omega/4\gamma_0^2 \end{cases} \quad (27)$$

The latter regime, being more rapid, is of greater interest here, and we focus on this instability in what follows. It is easily demonstrated that the linearised equations, allowing only k_y modes, can be reduced to the following set of coupled equations,

$$\begin{aligned} \frac{\partial v_x}{\partial t} &= \frac{\Omega}{2} v_0 \frac{n_1}{n_0} & , & \quad \frac{\partial v_y}{\partial t} = \Omega v_0 b_x, \\ \frac{\partial}{\partial t} \frac{n_1}{n_0} &= -c \frac{\partial v_y}{\partial y} & , & \quad \frac{\partial b_x}{\partial t} = \frac{c}{2} \frac{\partial v_x}{\partial y}. \end{aligned} \quad (28)$$

Motion in the z -direction can be neglected at short wavelengths. Considering a linearly polarised wave $\delta B_x(y)$, it is readily seen that the thermal protons are deflected in the y -direction by the Lorentz force, resulting in compressions and rarefactions along the y -axis. This bunching produces a perturbed current $\delta j_z = en_1 v_0$ which accelerates plasma in the x direction. The resulting plasma motions have negative and positive regions of shear flow (vorticity) in regions of negative and positive B_x respectively, which results in further growth of magnetic field, causing a runaway instability.

The similarity between this instability and the well known Rayleigh-Taylor instability is self-evident. In particular, recalling the acceleration of the non-inertial frame, $g = cv_0\Omega$, the growth rate can be written in the more familiar form:

$$\omega = i\sqrt{\frac{gk}{2}} \quad (29)$$

The instability for perpendicular shocks can thus be understood in terms of a Rayleigh-Taylor instability, due to the effective gravity introduced by the zeroth order $\beta_0 \times \mathbf{B}_0$ force. The linear analysis presented above, will clearly break down for $n_1 > n_0$, however, further mixing/penetration of

high-density spikes, will allow continued non-linear growth of the field, should sufficient time be available.

Having identified the relevant instabilities that can operate on long wavelengths at ultra-relativistic shocks, we now address the issue of whether sufficient time is available to allow significant field growth on the scales of interest, as this is most crucial when it comes to cosmic-ray acceleration.

5 APPLICATION TO COSMIC-RAY ACCELERATION

The acceleration of UHECRs requires growth of magnetic field fluctuations to a sufficient level and most importantly on a sufficient scale. If this is to be seeded in the upstream plasma, the instabilities described above, as a minimum requirement, must undergo at least one e-folding time. As mentioned previously, the size of the precursor will depend sensitively on the particle scattering, since it is only necessary to deflect the particle through an angle $\sim 1/\Gamma_{\text{sh}}$ (measured in the upstream frame), before the shock quickly overtakes it. We adopt a simple picture for the details of energetic particle deflections in the shock precursor, a more detailed discussion of the particle scattering behaviour can be found in Achterberg et al. (2001) or Couch et al. (2008). Note, however, that all calculations should be made in the frame in which the zero-th order electric field vanishes, which is different from previous authors.

5.1 Parallel shocks

As mentioned above, for a parallel (or indeed any subluminal) shock, in the absence of any pre-existing turbulence, the particles can outrun the shock indefinitely. However, if a small level of fluctuating magnetic field is introduced, $\delta B \ll B_0$, the particles can scatter/deflect on these perturbations, and eventually recross the shock. The fastest growing modes grow at wavelengths close to resonance with the thermal protons' Larmor radius, as measured in the electron drift frame, i.e. the protons have bulk Lorentz factor $\gamma_p = \sqrt{2\eta}\Gamma_{\text{sh}}$, and Larmor radius $\gamma_p mc^2/eB_0$. Note that B_0 is parallel to the shock normal, and hence invariant to boosts along this direction. The cosmic-ray Larmor radius is expected to be orders of magnitude larger than that of the thermal protons, such that the growth of modes at resonant wavelengths with the cosmic rays is too slow to have any significant effect, as is evident from inspection of Figure 2. Thus, resonant scattering is negligible unless non-linear effects can cause fields to grow to larger scales, as has been demonstrated to occur in simulations of non-relativistic shocks (Reville & Bell 2013; Bell et al. 2013). This requires on the order of 5–10 e-folding times in the upstream plasma, before a fluid element is overtaken by the shock.

We consider a simple pitch-angle diffusion model for the cosmic rays upstream of the shock, continuing to work in the electron drift frame. If the waves are initially on a scale $\lambda \ll r_{g,\text{cr}}$, a crude approximation to the pitch angle diffusion coefficient is

$$\mathcal{D}_\theta = \frac{\langle \Delta\theta^2 \rangle}{2\tau_{\text{sc}}} \approx \left(\frac{\delta B_\perp}{B_0} \right)^2 \frac{c\lambda}{r_{g,\text{cr}}^2}$$

If the cosmic ray, entering the upstream with velocity $\beta_z(0)$

normal to the shock, undergoes a series of small angle deflections, we can approximate the average deceleration as

$$\beta_z(t) \approx \beta_z(0) \left[1 - \frac{1}{2}\vartheta^2 \right] \sim \beta_z(0) [1 - \mathcal{D}_\theta^2 t] \quad (30)$$

The maximum precursor size is determined from the condition $\beta_z(t) - \beta_{\text{sh},e} = 0$, which in the limit $\beta_z(0) = 1$ is

$$L_{\text{pre}} \approx \frac{r_{g,\text{cr}}^2}{\lambda \gamma_{\text{sh}}^4} \left(\frac{B_0}{\delta B_\perp} \right)^2$$

The corresponding maximum growth time for streaming instabilities to develop is $t_{\text{grow}} = L_{\text{pre}}/\beta_{\text{sh},e}c$.

Inserting the maximum growth rate from equation (25), with $\lambda = 2\pi/k_{\text{max}}$, using the values for γ_p , γ_{sh} and γ_{cr} given in section 3, it follows that

$$\begin{aligned} \omega_{\text{max}} t_{\text{grow}} &\approx \frac{3}{4\pi} \frac{\gamma_{\text{cr}}^2}{\gamma_p^2 \gamma_{\text{sh},e}^4} \left(\frac{B_0}{\delta B_\perp} \right)^2 \\ &\approx (1 - 8\eta)^{-1/2} \left(\frac{B_0}{\delta B_\perp} \right)^2 \end{aligned} \quad (31)$$

which, for $\eta \ll 1$, is essentially independent of the parameters of the reflected component. For field growth to extend to larger scales, this number must exceed unity. However, the growth of magnetic field on small scales appears to self-limit the growth of the instability, since the number of e-foldings decreases rapidly as δB increases, implying that magnetic field amplification is limited to $\delta B \lesssim B_0$ and to scales $\ll r_{g,\text{cr}}$.

We again note that particles that cross the shock multiple times, with higher energies, can extend further into the upstream region, and allow for larger growth. As mentioned previously, the fraction of the energy η carried by these particles will be sensitive to the details of the particle spectrum. There are suggestions from Monte Carlo or PIC simulations, that a non-thermal tail $dn/d\gamma \propto \gamma^{-2.2}$ is produced (Achterberg et al. 2001; Sironi & Spitkovsky 2011), although other report considerably steeper spectra (Ostrowski & Bednarz 2002; Summerlin & Baring 2012). As the energy density in the beam decreases, the growth rate reduces to that found in Reville et al. (2006), which can have a larger maximum growth-rate, but at increasingly smaller length scales.

Parallel shocks represent a singular case, since the problems of maximum energy associated with the mean field transporting the highest energy particles into the downstream do not arise. While this does not rule out the possibility of multiple shock crossings, and acceleration to higher energies, it appears impossible to enter a regime where resonant scattering can occur at high energies. The problem thus becomes a time limitation at high energies. The isotropisation time \mathcal{D}_θ^{-1} , for UHECRs will become longer than the age of any realistic system. Similar conclusions were drawn by Sironi et al. (2013), for acceleration in purely small-scale fields. Adopting the Blandford-McKee solution for a relativistic expanding blast-wave (Blandford & McKee 1976), they found a maximum energy $\sim 10^{17}$ eV, in the ISM frame.

5.2 Perpendicular shocks

For perpendicular shocks, the situation is more straightforward, as to lowest order, particles simply undergo a regular

deflection in the upstream field, as measured in the electron frame i.e. the frame in which the zeroth order electric field vanishes. Following the same argument as above, the precursor length is

$$L_{\text{pre}} = \frac{r_{\text{g,cr}}}{\gamma_{\text{sh}}} \quad (32)$$

and maximum growth time as before. Using the shorter wavelength solution in (27) the number of e-foldings at a given wavenumber is

$$\omega t_{\text{grow}} \approx \sqrt{\frac{\Omega}{2} c k_y v_0} \frac{r_{\text{g,cr}}}{c \gamma_{\text{sh}}^3} \quad (33)$$

Since the growth at longer wavelengths is slower, this can be considered as an upper limit to the growth in general. Since the number of e-foldings increases indefinitely with wavenumber, k , it suffices to determine the wavelength at which one e-folding is achieved. This occurs at

$$k r_{\text{g,cr}} = \frac{2}{v_0} \gamma_{\text{sh}}^6 \frac{\gamma_0}{\gamma_{\text{cr}}} \approx \frac{1}{(8\eta)^2} \frac{\Gamma_{\text{sh}}^2}{\Gamma_{\text{cr}}} \quad (34)$$

where, as usual, lower case γ s refer to quantities measured in the electron rest frame. At the injection energy $\Gamma_{\text{cr}} \sim \Gamma_{\text{sh}}^2$, clearly $k r_{\text{g,cr}} \gg 1$, i.e. non-linear growth can only take place on wavelengths much less than gyro-radius of the cosmic rays. At higher energies, $\Gamma_{\text{cr}} \gg \Gamma_{\text{sh}}^2$, this value will depend on the shape of the spectrum. However, most results to date put the shape of the spectrum to be steeper than $dn/d\gamma \propto \gamma_{\text{cr}}^{-2.2}$. Thus $\eta \propto \gamma_{\text{cr}}^{-1.2}$ or steeper, such that the scales that one can expect non-linear growth, and the scale of the Larmor radius diverge with increasing energy. We note that since k lies along the field, and is linearly polarised in the plane orthogonal to the shock normal, and magnetic field, the length and relative magnitude, with respect to the mean field, are preserved. Since it is scattering in the downstream that determines whether a particle is advected downstream or not, it is the gyroradius of the cosmic ray in this frame that is most important. The gyro radius of a cosmic ray with Lorentz factor γ_{cr} in the electron frame has, on crossing into the downstream, a Larmor radius

$$r_{\text{g,d}} = \sqrt{8\eta} \gamma_{\text{cr}} \frac{mc^2}{e\bar{B}_{\text{d}}} = 8\eta \gamma_{\text{d}} \frac{mc^2}{e\bar{B}_{\text{d}}} \quad (35)$$

where \bar{B}_{d} is the shock compressed mean field. It follows that

$$k r_{\text{g,d}} \approx \frac{1}{8\eta} \frac{\Gamma_{\text{sh}}^2}{\Gamma_{\text{cr}}} \quad (36)$$

which diverges less quickly, but still exceeds unity. Since it is not possible to achieve even one e-folding close to gyro-resonance, the scattering waves for high-energy particles, close to the values given in equation (6) are not seeded upstream, via the plasma instabilities found in this paper. It is interesting to note that it is still possible to have several e-foldings on shorter wavelengths, which may allow for more rapid acceleration of ‘lower’ energy electrons (lower than the equations (6) or (7) limit).

Perpendicular shocks also offer another possibility for growth, due to either large-scale fluctuations pre-existing in the ambient medium, or non-uniform injection of particles over the shock surface. The former case will preserve the relative amplitude of the mean field with the perpendicular fluctuations, and as such, it has a negligible effect by itself,

unless the magnetic field threading the ambient medium is highly non-uniform $|\delta\mathbf{B}| \gg \langle\mathbf{B}\rangle$, e.g. for interaction with a striped wind. This would likely introduce a characteristic length/energy scale, which is disfavoured by the observations, which typically have power-law form (Band et al. 1993).

The latter case, of non-uniform injection over the shock surface can be treated in a reduced model. It is clear from the analysis in section 4.2, the return current is provided by the drift of the background protons in the electron frame, which previously was used to determine the acceleration of the non-inertial frame

$$\frac{d\mathbf{p}_{\text{p}}}{dt} = e\beta_0 \times \mathbf{B}_0 \quad (37)$$

where B_0 is the compressed value measured in the electron drift frame, described in section 3. If the injection is non-uniform, \mathbf{B}_0 varies in the plane of the shock, and neighbouring fluid elements will be differentially accelerated. The field will therefore shear, leading to amplification. Assuming the perpendicular velocity does not become relativistic, the displacement of neighbouring field points is

$$\frac{d^2\xi}{dt^2} \approx \frac{eB_{\perp}(\mathbf{r})}{\gamma_{\text{p}} m_{\text{p}}} \beta_0 \quad (38)$$

The maximum possible separation of neighbouring fluid element in the precursor can be estimated assuming neighbouring regions are at the two extremes $\eta_1 \Gamma_{\text{sh}} \gg 1$, and $\eta_2 \Gamma_{\text{sh}} \approx 0$, in which case

$$\Delta\xi_{\text{max}} < \frac{\gamma_{\text{cr}}}{\gamma_{\text{p}} \gamma_{\text{sh}}^6} r_{\text{g,cr}}$$

where all terms are again measured in the electron drift frame. Since this perturbation is perpendicular to both the shock normal and mean field, it is preserved on crossing the shock. Using the values from section 3, and the above expression for $r_{\text{g,d}}$, it follows that

$$\frac{\Delta\xi_{\text{max}}}{r_{\text{g,d}}} < \eta \frac{\Gamma_{\text{cr}}}{\Gamma_{\text{sh}}^2} \quad (39)$$

which, modulo a factor of order unity, is essentially equivalent to the equation (36), which has already been demonstrated to be insufficient.

5.3 Instabilities downstream of the shock

The above instabilities, occurring in the foreshock region, are insufficient to provide effective scattering of high energy particles. It is possible, however, that currents downstream of the shock can excite plasma instabilities. It is well known that the distribution of cosmic rays is highly anisotropic at relativistic shocks. As measured in the downstream frame, the distribution function is peaked at pitch angles closely aligned with the plane of the shock (Kirk et al. 2000).

For all but a small fraction ($\sim 1/\Gamma_{\text{sh}}$) of possible field configurations, the magnetic field downstream of the shock lies in the plane of the shock. The anisotropic distribution thus produces a current, which we can take to orthogonal to both the shock normal and the magnetic field. The situation is therefore similar to the perpendicular field case considered above, only now the background plasma is relativistically hot, with mean temperature $\Gamma_{\text{sh}} mc^2$. The magnitude of the cosmic-ray current will depend on the degree

of anisotropy, but for solutions given in Kirk et al. (2000), immediately downstream of the shock it is a sizeable fraction of the speed of light (~ 0.5). This current will persist in the downstream until it has been isotropised, which is again determined by the diffusion coefficient D_θ . Assuming scattering is dominated by deflections in small-scale Weibel structures, the current exists in the downstream frame for a time

$$t_{\text{grow}} \sim D_\theta^{-1} \quad (40)$$

To calculate the other relevant parameters, we assume the cosmic rays are overtaken by the shock without significant scattering and with pitch angle $\theta \gtrsim 1/\Gamma_{\text{sh}}$. Making a Lorentz transform into the downstream frame results in a reduction by a factor $\sim \Gamma_{\text{sh}}$ of both the average energy per particle as well as the number density of the cosmic-rays. The background protons, on the other hand, are compressed, with proper compression ratio $\sqrt{8}\Gamma_{\text{sh}}$. The total number density of cosmic rays in the upstream is determined by particles at the injection energy $\Gamma_{\text{sh}}^2 mc^2$, such that in the downstream $\bar{n}_{\text{cr}} \sim \eta \bar{n}_{\text{p}}$, and decreasing as before for higher energies.

If the electrons are also thermalised in the downstream proton frame, the background plasma can be treated as a single fluid, and we can take as an upper limit on the acceleration of a Lagrangian fluid element, with all quantities are measured in the downstream frame,

$$\frac{d^2\xi}{dt^2} < \frac{en_{\text{cr}}\beta_{\text{drift}}\langle B_d \rangle}{\Gamma_{\text{sh}}m_{\text{p}}n_{\text{p}}} \sim \eta\beta_{\text{drift}}\frac{e\langle B_d \rangle}{\Gamma_{\text{sh}}m_{\text{p}}} \quad (41)$$

where $\langle B_d \rangle$ is the shock compressed mean field $\sim \sqrt{8}\Gamma_{\text{sh}}B_\perp^0$.

Using the expression for D_θ given in equation (4), and taking the parameters on the right hand side of (41) not to vary in time, the maximum displacement is

$$\frac{\Delta\xi_{\text{max}}}{r_{\text{g,d}}} < \frac{\eta\beta_{\text{drift}}}{\Gamma_{\text{sh}}^2} \frac{\sigma_u}{\sigma_d} \left(\frac{\gamma_d}{\Gamma_{\text{sh}}}\right)^3 \left(\frac{c/\omega_{\text{pp}}}{\lambda}\right)^2 = \frac{\eta\beta_{\text{drift}}}{\gamma_{\text{d,max}}^2} \left(\frac{\gamma_d}{\Gamma_{\text{sh}}}\right)^3$$

where $r_{\text{g,d}}$ is the cosmic-ray gyro radius measured in the mean field, as in the previous section, and $\gamma_{\text{d,max}}$ is as given in equation (6). Again, this is insufficient at low energies $\gamma_d \sim \Gamma_{\text{sh}}$, and does not improve at higher energies, on account of η .

6 DISCUSSION

Ultra-relativistic shocks are frequently identified as, or associated with strong sources of non-thermal radiation. Shock acceleration offers a well-tested mechanism for generating the non-thermal particle populations required to produce this emission, and theory and simulations are converging to provide a more complete picture of the plasma processes occurring in these environments. From a phenomenological perspective, observations can typically be explained using simple leptonic models, whereas extracting information about non-thermal hadronic populations is challenging. However, on theoretical grounds, it is expected that any source capable of accelerating protons or nuclei to ultra-high energies $> 10^{18}$ eV, is likely also to accelerate electrons rapidly, at least to their radiation reaction limit. This motivates consideration of strong gamma-ray emitters, such as GRB, pulsars and AGN as potential candidates for

UHECR production, all of which are thought to contain ultra-relativistic shocks.

Motivated by recent successes in numerical modelling of the micro-physics of relativistic shocks, the acceleration of particles at weakly-magnetised ultra-relativistic shocks has been investigated. We demonstrate that Weibel-mediated shocks, or indeed any shock mediated by kinetic instabilities, operating at the plasma-skin depth, cannot accelerate particles above a critical energy, given in equation (7) (see also Lemoine & Pelletier 2010):

$$E_{\text{max}} \approx \left(\frac{\Gamma_{\text{sh}}}{100}\right)^2 \left(\frac{\lambda_d}{10c/\omega_{\text{pp}}}\right) \left(\frac{\sigma_d}{10^{-2}}\right) \left(\frac{\sigma_u}{10^{-8}}\right)^{-1/2} \text{ PeV},$$

Acceleration to higher energies can only be achieved if strong scattering can occur on scales $\gg c/\omega_{\text{pp}}$. We have investigated the growth of plasma instabilities, driven by cosmic-ray currents, both upstream and downstream of an ultra-relativistic shock in this long-wavelength limit. In all cases, it is demonstrated that the growth of any such instability is too slow, on the scales required to facilitate acceleration to higher energies. As such, our results suggest that the above energy limit can not be circumvented, implying this maximum energy is an inherent limitation of shock-acceleration at ultra-relativistic shocks. Future simulations that can self-consistently investigate the scattering of ultra-relativistic particles in self-generated relativistic-plasma turbulence may ultimately be required to provide confirmation.

We emphasise that it is not suggested that the instabilities presented in this paper do not play any role. In fact, they may still lead to significant growth of magnetic field on scales larger than the ion collisionless skin depth, which may increase the rate of acceleration for lower energy particles. This may be an essential feature in the case of electron acceleration at GRB shocks, in the presence of radiation reaction limited acceleration (Kirk & Reville 2010).

While this paper is by no means the first to suggest that GRBs are not the source of UHECRs (e.g. Milosavljević & Nakar 2006), we have gone a step further, in demonstrating that ultra-relativistic shocks are dis-favoured as sources of high energy particles in general. The maximum energy, given above, for typical parameters expected in external GRB shocks, for example, may have a detectable signature, which is measurable with the next generation Cherenkov observatory, CTA (Acharya et al. 2013).

Processes other than shock acceleration may provide additional mechanisms for UHECR production in ultra-relativistic jets (e.g. Ostrowski 1998; Rieger & Duffy 2005), or around rapidly rotating compact objects (e.g. Bell 1992; Fang et al. 2013). However, the acceleration mechanisms operating in such flows are not near as well established as the Fermi shock acceleration process, although this may change with time. Diffusive shock acceleration at large-scale non-relativistic shocks, such as those expected to be found in galaxy mergers, or AGN radio ‘hot-spots’, for example, offer alternative, highly plausible candidates.

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APPENDIX A: KINETIC TREATMENT FOR PARALLEL SHOCK CASE

The dispersion relation for circularly polarised waves propagating parallel to the mean field is (e.g. Reville et al. 2006)

$$\frac{k_{\parallel}^2 c^2}{\omega^2} - 1 = \sum_j \chi_j \quad (\text{A1})$$

where

$$\chi_j = \Gamma_j \frac{\omega_{pj}^2}{\omega^2} \int \frac{d^3 u}{\gamma} f_j(\vec{u}) \left[\frac{-\omega\gamma + ck u_{\parallel}}{D(u_{\parallel})} - \frac{u_{\perp}^2}{2} \frac{(c^2 k^2 - \omega^2)}{D^2(u_{\parallel})} \right]$$

is the susceptibility for each plasma component. We work with the normalised 4-momentum $u^{\mu} = dx^{\mu}/ds$, and $\omega_{pj}^2 = 4\pi n_j q_j^2 / \Gamma_j m_j$ and $\omega_{cj} = q_j B_0 / m_j c$ are the relativistic plasma frequency and gyro-frequency for each species j . Here, $n_j, \Gamma_j = \int (1, \gamma) f_j(\mathbf{u}) d^3 u$ is the mean density, Lorentz factor of each species. The resonant denominator is

$$D(u_{\parallel}) = \varepsilon \omega_{cj} (1 + Z(u_{\parallel})) \quad (\text{A2})$$

with

$$Z(u_{\parallel}) = \frac{\omega\gamma - ck u_{\parallel}}{\varepsilon \omega_{cj}} \quad (\text{A3})$$

and the waves have left(right)-handed polarisation for $\varepsilon = +1(-1)$, for $k > 0$. It is readily noticed that $\omega^2 \chi_j$ is an invariant quantity, allowing us to calculate the dispersion relation in an arbitrary frame.

We consider a three component plasma, protons, electrons and cosmic-rays (also protons), all components being treated as cold beams

$$f_j(u_{\parallel}, u_{\perp}) = \frac{1}{2\pi u_{\perp}} \delta(u_{\perp}) \delta(u_{\parallel} - \Gamma_j \beta_j) \quad (\text{A4})$$

The susceptibility of each component is thus

$$\omega^2 \chi_j(k, \omega) = \frac{\omega_{pj}^2 \Gamma_j (c\beta_j k - \omega)}{\varepsilon \omega_{cj} - \Gamma_j (c\beta_j k - \omega)} \quad (\text{A5})$$

Assuming the electrons are magnetised in their own rest frame, $\Gamma_e |\omega - c\beta_e k| \ll |\omega_{ce}|$, the denominator of the electron susceptibility can be expanded to give,

$$\omega^2 \chi_e = \Gamma_e \frac{\omega_{pe}^2}{\varepsilon \omega_{ce}} (c\beta_e k - \omega) + \Gamma_e^2 \frac{\omega_{pe}^2}{\omega_{ce}^2} (c\beta_e k - \omega)^2 + \dots \quad (\text{A6})$$

Working from this point on exclusively in the electron rest frame ($\beta_e = 0$), using subscript b to represent the cosmic-ray protons, the full dispersion relation reads

$$k^2 c^2 - \left(1 + \frac{\omega_{pe}^2}{\omega_{ce}^2}\right) \omega^2 + \frac{\omega_{pe}^2}{\varepsilon \omega_{ce}} \omega + \quad (\text{A7})$$

$$\frac{\omega_{pb}^2 \Gamma_b (\omega - c\beta_b k)}{\varepsilon \omega_{cb} + \Gamma_b (\omega - c\beta_b k)} + \frac{\omega_{pb}^2 \Gamma_p (\omega - c\beta_p k)}{\varepsilon \omega_{cp} + \Gamma_p (\omega - c\beta_p k)} = 0$$

The roots of this equation are found numerically and are plotted in Figures 1 and 2.

The dispersion relation, equation (24) can also be derived from this equation, and helps demonstrate the limitations of the approximate approach used in the main section of the paper. Neglecting the first three terms on the left hand side of equation (A7), and taking the ultra-relativistic limit, $n_e = 2n_p = 2n_b$, in the limit of $\beta_b = -1$, the dispersion

relation for $\varepsilon = +1$ can be expressed as

$$\tilde{\omega}^3 + \frac{1+z}{2}\tilde{\omega}^2 + \tilde{k} \left[1 - \beta_p(z + \tilde{k}) \right] \tilde{\omega} + \frac{1+z}{2}\beta_p \tilde{k}^2 = 0 \quad (\text{A8})$$

where $z = \Gamma_p/\Gamma_b \ll 1$, and

$$\tilde{\omega} = \Gamma_p \omega / \omega_c \quad \text{and} \quad \tilde{k} = \Gamma_p k c / \omega_c .$$

In deriving (A8), all terms containing $1 - \beta_p$ have been neglected. An approximate solution to this cubic can be found by solving for \tilde{k} ,

$$\frac{\beta_p k}{\omega} = \frac{1 - \beta_p z \pm 2\beta_p^{1/2} \sqrt{\omega^2 - z}}{2\omega - (1+z)} \quad (\text{A9})$$

where we have dropped the tildes. For modes with $|\omega^2| \gg z$, the frequency for unstable modes is

$$\omega = \frac{1}{2} \left(\beta_p k - \frac{1 - z\beta_p}{2} \right) \pm \frac{1}{2} \sqrt{\left(\beta_p k - \frac{1 - z\beta_p}{2} \right)^2 + 2z - 2(1+z)\beta_p k} \quad (\text{A10})$$

which clearly reduces to (24) in the limit $z \rightarrow 0$.